Some Notes on Generalized Young Inequality for n Numbers

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ABSTRACT
In this note, we obtain a generalized of Young's inequality for n numbers. From the inequality, we also get generalized of Hölder's Inequality and Minkowski's inequality for n terms. Furthermore result, some improvements of the generalized Young's inequality for n numbers are discussed.

Keywords: *Young's inequality, Holder's inequality, Minkowski's inequality.*

1. INTRODUCTION

Let a and b be positive numbers. The Famous Young inequality [1] states that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \#(1.1)$$

for every $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ or $q = \frac{p}{p-1}$.

The Classical Young inequality is also rewritten as

$$a^{\lambda} b^{1-\lambda} \leq \lambda a + (1-\lambda) b \#(1.2)$$

by putting $a := a^{\frac{1}{p}}$, $b := b^{\frac{1}{q}}$, and $\lambda = \frac{1}{p}$ (clearly, $0 \leq \lambda \leq 1$).

Based on Young's inequality, we can derive two other well-known inequalities namely Holder's inequality and Minkowski's inequality, which are some applications of Young's inequality. Apart from these two inequalities, Young's inequality also has many other applications. So, many mathematicians are interested in discussing Young's inequality.

Many researchers also try to generalize, improve, and refine these inequalities. An improvement of Young's inequality, obtained by F. Kittaneh and Y. Manasrah [2], is as follows

$$a^{\lambda} b^{1-\lambda} + r_0 \left(\sqrt{a} - \sqrt{b}\right)^2 \leq \lambda a + (1-\lambda) b \#(1.3)$$

with $r_0 = \{\lambda, 1-\lambda\}$.

The authors of [3] obtained another refinement of the Young inequality as

$$\left(a^{\lambda} b^{1-\lambda}\right)^2 + r_0^2 (a-b)^2 \leq (\lambda a + (1-\lambda) b)^2 \#(1.4)$$

with $r_0 = \{\lambda, 1-\lambda\}$.

Other development results of Young's inequality can also be seen in [4-9].

On the other hand, we know that the Young's inequality for two scalars is a fundamental relation between two non-negative real numbers, which is the generalized arithmetic-geometric mean inequality with weight. The question arises: what is the generalization of Young's inequality if there are n numbers? as an application of the generalization of these inequalities, what inequalities can be generated? And what is the form of development of the generalization of Young's inequality?

In this Article, we will present the generalization of Young's inequality involving n numbers. This paper will also discuss the inequalities that will result from the generalization of these inequalities, in particular the generalization of Hölder's inequality and generalization of Minkowski's inequality. Furthermore, inspired by the refinement of [2] and [3], we will also discuss the refinement of the generalization of these inequality.

2. MAIN RESULTS

2.1. Generalized Young's inequality for n numbers

We will start by introducing a generalization of Young's inequality involving n numbers. This form is a generalization of the classical form of Young's inequality.

Theorem 1. If a_i are positive numbers for all $i = 1,2,3,\dots,n$ and $p_i > 1$ for all $i = 1,2,3,\dots,n$ such that $\sum_{k=1}^{n} \frac{1}{p_k} = 1$, then

$$\prod_{i=1}^{n} a_i \le \sum_{i=1}^{n} \frac{1}{p_i}(a_i)^{p_i} \#(2.1)$$

Proof: We will apply mathematical induction to prove this theorem. For $n = 1$, the inequality is clearly satisfied. For $n = 2$, the inequality is directly satisfied from Young's inequality.

Now, assume that the inequality is true for $n = k$. we get

$$\prod_{i=1}^{k} a_i \le \sum_{i=1}^{k} \frac{1}{p_i}(a_i)^{p_i}$$

The next, we will proof that the inequality is also true for $n = k + 1$.

$$\begin{aligned}\prod_{i=1}^{k+1} a_i &= a_1 \cdot a_2 \cdot \dots \cdot a_{k-1} \cdot (a_k a_{k+1}) \\ &\le \sum_{k=1}^{k-1} \frac{1}{p_i}(a_i)^{p_i} + \frac{1}{p_k}(a_k a_{k+1})^{p_k} \\ &= \sum_{k=1}^{k-1} \frac{1}{p_i}(a_i)^{p_i} \\ &+ \frac{1}{p_k}(a_k)^{p_k}(a_{k+1})^{p_k} \\ &\le \sum_{k=1}^{k-1} \frac{1}{p_i}(a_i)^{p_i} \\ &+ \frac{1}{p_k}\left[\left(\frac{((a_k)^{p_k})^{\frac{q}{p_k}}}{\frac{q}{p_k}}\right) \right. \\ &\left. + \left(\frac{((a_{k+1})^{p_k})^{\frac{q}{q-p_k}}}{\frac{q}{q-p_k}}\right)\right] \\ &= \sum_{k=1}^{k+1} \frac{1}{q_i}(a_i)^{q_i}\end{aligned}$$

where $q_i = p_i$ for $i = 1,\dots,k-1$, $q_k = q$, and $q_{k+1} = \frac{p_k q}{q-p_k}$. Furthermore, we can easily prove that $\sum_{k=1}^{k+1} \frac{1}{q_i} = 1$. So, the inequality has been proven. ■

Next, the inequality (2.1) can be written as

$$\prod_{i=1}^{n} (a_i)^{\lambda_i} \le \sum_{i=1}^{n} \lambda_i a_i \#(2.2)$$

Remark. When comparing the inequality (2.1) with the inequality (1.1), or (1.2) with (2.2), it is clearly seen that the left-hand side and the righthand side in the inequality (2.1) or (2.2) consist of n numbers, while in the inequality (1.1) or (1.2) there are only 2 numbers. It should be noticed here that either the inequality (2.1) or (2.2) is a generalization of the inequality (1.1) or (1.2).

2.2. Generalized Hölder's inequality with n terms

In this chapter, we will present a generalization of H ö lder's inequalitiy obtained from the generalized Young's inequalities for n numbers.

Theorem 2.2. For any vectors x_i in C^m, and for any positif numbers p_i satisfying $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, we have

$$\sum_{k=1}^{m} \left|\prod_{i=1}^{n} x_{i_k}\right| \le \prod_{i=1}^{n} \|x_i\|_{p_i}, \#(2.3)$$

where

$$\|x_i\|_{p_i} = \left(\sum_{k=1}^{m} |x_{i_k}|^{p_i}\right)^{\frac{1}{p_i}}.$$

Proof: If one of x_i is zero, the inequality certainly holds with equality. Otherwise, assume x_i are nonzero for every $i \in \{1,2,3,\dots,n\}$, and let $u_i = \frac{x_i}{\|x_i\|_{p_i}}$, and note that $\|u_i\|_{p_i} = 1$ for all i. Then by using equation (2.1)

$$\begin{aligned}\sum_{k=1}^{m} \left|\prod_{i=1}^{n} u_{i_k}\right| &= \sum_{k=1}^{m} \left(\prod_{i=1}^{n} |u_{i_k}|\right) \\ &\le \sum_{k=1}^{m} \left(\sum_{i=1}^{n} \frac{1}{p_i}|u_{i_k}|^{p_i}\right) \\ &= \sum_{i=1}^{n} \left(\frac{1}{p_i}\left(\|u_i\|_{p_i}\right)^{p_i}\right) = 1.\end{aligned}$$

Now multiplying both sides by the positive quantity $\prod_{i=1}^{n} \|x_i\|_{p_i}$ to obtain the statement of the theorem. To achieve equality, each term in the sum must achieve equality in inequality (2.1), i.e., for all $k \in \{1,2,3,\dots,m\}$,

$|u_{1_k}| = |u_{2_k}| = \cdots = |u_{n_k}|$, which translates to the statement in the theorem since $|u_i| = \frac{|x_i|}{\|x_i\|_{p_i}}$ for all i. ■

Remark. The inequality (2.3) is a generalization of the Hölder inequality which is known as follows. For any vectors x and y in C^m, and for any positive numbers p and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\sum_{k=1}^{m} |x_k + y_k| \leq \|x\|_p \|x\|_q,$$

Where $\|x\|_p = (\sum_{k=1}^{m} |x_k|^p)^{\frac{1}{p}}$ dan $\|y\|_p = (\sum_{k=1}^{m} |y_k|^q)^{\frac{1}{q}}$. We also have known that Holders's inequality also has many other applications in normed spaces ([10], [11], [12], [13]).

2.3. Generalized Minkowski's inequality with n terms

In this chapter, we will also introduce a generation of Minkowski's inequalities.

Theorem 2.3. For any vectors u_i in C^m, and for any positive number $p > 1$, we have

$$\|\sum_{i=1}^{n} u_i\|_p \leq \sum_{i=1}^{n} \|u_i\|_p. \#(2.4)$$

Equality hold if and only if $au_i = bu_j$ for every $i \neq j$ and for some non-negative real constants a and b, not both zero.

Proof:

$$\left(\|\sum_{i=1}^{n} u_i\|_p\right)^p = \sum_{k=1}^{m} \left|\sum_{i=1}^{n} u_{i_k}\right|^p$$

$$= \sum_{k=1}^{m} \left(\left|\sum_{i=1}^{n} u_{i_k}\right| \cdot \left|\sum_{i=1}^{n} u_{i_k}\right|^{p-1}\right)$$

$$\leq \sum_{k=1}^{m} \left(\left(\sum_{i=1}^{n} |u_{i_k}|\right) \cdot \left|\sum_{i=1}^{n} u_{i_k}\right|^{p-1}\right)$$

$$= \sum_{i=1}^{n} \left(\sum_{k=1}^{m} \left(|u_{i_k}| \left|\sum_{i=1}^{n} u_{i_k}\right|^{p-1}\right)\right)$$

$$\leq \sum_{i=1}^{n} \left(\sum_{k=1}^{m} |u_{i_k}|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{m} \left(\left|\sum_{i=1}^{n} u_{i_k}\right|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$$

$$= \left(\sum_{i=1}^{n} \|\sum_{i=1}^{n} u_i\|_p\right) \left(\|\sum_{i=1}^{n} u_i\|_p\right)^{p-1}.$$

The theorem follows by dividing both sides by the positive quantity $\left(\|\sum_{i=1}^{n} u_i\|_p\right)^{p-1}$. To achieve equality it is necessary that the triangle inequality (n numbers) for complex numbers holds with equality for each term. ■

Remark. The inequality (2.4) is a generalization of the Minkowski's inequality which is known as follows. For any vectors u and v in C^m, and for any positive number $p > 1$, we have

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p.$$

Equality hold if and only if $au = bv$ for some non-negative real constants a and b, not both zero. We also have known that Minkowski's inequality also has many other applications ([14])

3. REFINEMENTS OF THE SCALAR YOUNG INEQUALITY

In this chapter, we will give some refinements of Generalized Young's Inequality.

Theorem 3.1. If a_i are positive numbers for all $i = 1,2,3,\ldots,n$, $0 \leq \lambda_i \leq 1$ for all $i = 1,2,3,\ldots,n$ such that $\sum_{i=1}^{n} \lambda_i = 1$, and $s \in N$ with $1 \leq s \leq n$, then

$$\prod_{i=1}^{n} a_i^{\lambda_i} + \sum_{i=1\, i\neq s}^{n} r_i\left(\sqrt{a_i} - \sqrt{a_s}\right)^2 \leq \sum_{i=1}^{n} \lambda_i a_i \#(3.3)$$

where $r_i = \{\lambda_i, 1 - \lambda_i - 2\sum_{j=1\, j\neq i,s}^{n} \lambda_j\}$ for $i \in \{1,2,3,\ldots,n\}$.

Proof: If $\sum_{i=1\, i\neq s}^{n} \lambda_i = \frac{1}{2}$ for every i and $n = 2$, the inequality becomes an equality. Assume that $\sum_{i=1\, i\neq s}^{n} \lambda_i < \frac{1}{2}$. Then $\lambda_i < \frac{1}{2}$ for every $i \in \{1,2,3,\ldots,n\} - \{s\}$ and by inequality (2.1), we have

$$\sum_{i=1}^{n} \lambda_i a_i - \sum_{i=1\ i\neq s}^{n} \lambda_i\left(\sqrt{a_i}-\sqrt{a_s}\right)^2 = \sum_{i=1}^{n} \lambda_i a_i - \sum_{i=1\ i\neq s}^{n} \lambda_i\left(a_i - 2\sqrt{a_i a_s} + a_s\right) = \sum_{i=1\ i\neq s}^{n} 2\lambda_i\sqrt{a_i a_s} + \left(1-2\sum_{i=1\ i\neq s}^{n}\lambda_i\right)a_s \geq \left(\prod_{i=1\ i\neq s}^{n}\left(\sqrt{a_i a_s}\right)^{2\lambda_i}\right)\cdot a_s^{\left(1-2\sum_{i=1\ i\neq s}^{n}\lambda_i\right)} = \left(\prod_{i=1\ i\neq s}^{n}(a_i)^{\lambda_i}\right)\cdot a_s^{\left(1-\sum_{i=1\ i\neq s}^{n}\lambda_i\right)} = \left(\prod_{i=1}^{n}(a_i)^{\lambda_i}\right),$$

where $\lambda_s = 1-\sum_{i=1\ i\neq s}^{n}\lambda_i$, and so

$$\sum_{i=1}^{n} \lambda_i a_i - \sum_{i=1\ i\neq s}^{n} \lambda_i\left(\sqrt{a_i}-\sqrt{a_s}\right)^2 \leq \prod_{i=1}^{n} a_i^{\lambda_i}.$$

If $\sum_{i=1\ i\neq s}^{n}\lambda_i > \frac{1}{2}$, let there is $i_0 \in \{1,2,\ldots,n\}-\{s\}$ such that $\lambda_{i_0} > \frac{1}{2}$, then $\lambda_i < \frac{1}{2}$ for every $i \in \{1,2,3,\ldots,n\}-\{i_0\}$ and by the inequality (2.1) we get

$$\sum_{i=1}^{n} \lambda_i a_i - \sum_{i=1\ i\neq i_0\ i\neq s}^{n} \lambda_i\left(\sqrt{a_i}-\sqrt{a_s}\right)^2 - \left(1-\lambda_{i_0} - 2\sum_{i=1\ i\neq i_0\ i\neq s}^{n}\lambda_i\right)\left(\sqrt{a_{i_0}}-\sqrt{a_s}\right)^2$$
$$= \sum_{i=1\ i\neq i_0}^{n} \lambda_i a_i - \sum_{i=1\ i\neq i_0\ i\neq s}^{n} \lambda_i\left(\sqrt{a_i}-\sqrt{a_s}\right)^2 + \lambda_{i_0}a_{i_0} - \left(1-\lambda_{i_0} - 2\sum_{i=1\ i\neq i_0\ i\neq s}^{n}\lambda_i\right)\left(\sqrt{a_{i_0}}-\sqrt{a_s}\right)^2$$
$$= \sum_{i=1\ i\neq i_0\ i\neq s}^{n} 2\lambda_i\sqrt{a_i a_s} + \left(2\sum_{i=1\ i\neq s}^{n}\lambda_i - 1\right)a_{i_0} + 2\left(1-\lambda_{i_0} - 2\sum_{i=1\ i\neq i_0\ i\neq s}^{n}\lambda_i\right)\sqrt{a_{i_0}a_s}$$
$$\geq \left(\prod_{i=1\ i\neq i_0\ i\neq s}^{n}\left(\sqrt{a_i a_s}\right)^{2\lambda_i}\right)\cdot a_{i_0}^{\left(2\sum_{i=1\ i\neq s}^{n}\lambda_i-1\right)}\cdot \sqrt{a_{i_0}a_s}^{2\left(1-\lambda_{i_0}-2\sum_{i=1\ i\neq i_0\ i\neq s}^{n}\lambda_i\right)}$$
$$= \left(\prod_{i=1\ i\neq i_0\ i\neq s}^{n}(a_i)^{\lambda_i}\right)\cdot\left(a_{i_0}\right)^{\lambda_{i_0}}\cdot(a_s)^{\left(1-\sum_{i=1\ i\neq s}^{n}\lambda_i\right)}. = \left(\prod_{i=1}^{n}(a_i)^{\lambda_i}\right),$$

Where $\lambda_s = 1-\sum_{i=1\ i\neq s}^{n}\lambda_i$, and so

$$\sum_{i=1}^{n} \lambda_i a_i - \sum_{i=1\, i\neq i_0\, i\neq s}^{n} \lambda_i\left(\sqrt{a_i}-\sqrt{a_s}\right)^2 - \left(1-\lambda_{i_0} - 2\sum_{i=1\, i\neq i_0\, i\neq s}^{n} \lambda_i\right)\left(\sqrt{a_{i_0}}-\sqrt{a_s}\right)^2 \leq \prod_{i=1}^{n} a_i^{\lambda_i}$$

Hence,

$$\prod_{i=1}^{n} a_i^{\lambda_i} + \sum_{i=1\, i\neq s}^{n} r_i\left(\sqrt{a_i}-\sqrt{a_s}\right)^2 \leq \sum_{i=1}^{n} \lambda_i a_i$$

This completes the proof. ■

As a direct consequence of Theorem 3.1, we have

$$\sum_{i_j\in\{1,2,3,\ldots,n\}\, i\neq k\to i_j\neq k_j} \left(\prod_{i=1}^{n} a_i^{\lambda_{i_j}}\right) + n\sum_{i=1\, i\neq s}^{n} r_i\left(\sqrt{a_i}-\sqrt{a_s}\right)^2 \leq \sum_{i=1}^{n} a_i$$

and so

$$\frac{1}{n}\sum_{i_j\in\{1,2,3,\ldots,n\}} \left(\prod_{i=1}^{n} a_i^{\lambda_{i_j}}\right) + \sum_{i=1\, i\neq s}^{n} r_i\left(\sqrt{a_i}-\sqrt{a_s}\right)^2 \leq \sum_{i=1}^{n} a_i.$$

Corollary 3.2. If a_i are positive numbers for all $i = 1,2,3,\ldots,n$ and $0 \leq \lambda_i \leq 1$ for all $i = 1,2,3,\ldots,n$ such that $\sum_{i=1}^{n} \lambda_i = 1$, then

$$\frac{1}{n}\sum_{i_j\in\{1,2,3,\ldots,n\}} \left(\prod_{i=1}^{n} a_i^{\lambda_{i_j}}\right) + \sum_{i=1\, i\neq s}^{n} r_i\left(\sqrt{a_i}-\sqrt{a_s}\right)^2 \leq \sum_{i=1}^{n} a_i \#(3.2)$$

Corollary 3.3. If a_i are positive numbers for all $i = 1,2,3,\ldots,n$ and $0 \leq \lambda_i \leq 1$ for all $i = 1,2,3,\ldots,n$ such that $\sum_{i=1}^{n} \lambda_i = 1$, then

$$\left(\prod_{i=1}^{n} a_i^{\lambda_i}\right)^2 + \sum_{i=1\, i\neq s}^{n} r_i(a_i-a_s)^2 \leq \sum_{i=1}^{n} \lambda_i a_i^2 < \left(\sum_{i=1}^{n} \lambda_i a_i\right)^2 \#(3.3)$$

where $r_i = \{\lambda_i, 1-\lambda_i - 2\sum_{j=1\, j\neq i,s}^{n-1} \lambda_j\}$ for $i \in \{1,2,3,\ldots,n\}$.

Proof:

If we putting a_i by a_i^2, the inequality can be written in the form

$$\left(\prod_{i=1}^{n} a_i^{\lambda_i}\right)^2 + \sum_{i=1}^{n-1} r_i(a_i-a_n)^2 \leq \sum_{i=1}^{n} \lambda_i a_i^2 < \left(\sum_{i=1}^{n} \lambda_i a_i\right)^2$$

■

Theorem 3.2. If a_i are positive numbers for all $i = 1,2,3,\ldots,n$ and $0 \leq \lambda_i \leq 1$ for all $i = 1,2,3,\ldots,n$ such that $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, then

$$\prod_{i=1}^{n} a_i^{\lambda_i} + \sum_{i=1}^{n-1} r_i\left(\sqrt{a_i}-\sqrt{a_{i+1}}\right)^2 \leq \sum_{i=1}^{n} \lambda_i a_i \#(3.3)$$

where $r_i = \sum_{k=1}^{i} (-1)^{k+1}\lambda_{i-k+1}$ for $i \in \{1,2,3,\ldots,n\}$.

Proof: If $\lambda_i = \frac{1}{2}$ for every i and $n = 2$, the inequality becomes an equality. Assume that $0 \leq \lambda_i \leq 1$ for all $i = 1,2,3,\ldots,n$ such that $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, then $\sum_{i=1}^{n-1} \lambda_k + (-1)^{k+1}\lambda_{n-k} < \frac{1}{2}$. Hence by the inequality (2.1), we have

$$\sum_{i=1}^{n} \lambda_i a_i - \sum_{i=1}^{n-1}\left(\sum_{k=1}^{i}(-1)^{k+1}\lambda_{i-k+1}\right)\left(\sqrt{a_i}-\sqrt{a_{i+1}}\right)^2$$
$$= \sum_{i=1}^{n}\lambda_i a_i - \sum_{i=1}^{n-1}\left(\sum_{k=1}^{i}(-1)^{k+1}\lambda_{i-k+1}\right)\left(a_i - 2\sqrt{a_i a_{i+1}} + a_{i+1}\right)$$
$$= \sum_{i=1}^{n-1} 2\left(\sum_{k=1}^{i}(-1)^{k+1}\lambda_{i-k+1}\right)\sqrt{a_i a_{i+1}} + \left(1-\sum_{i=1}^{n-1}\lambda_k + (-1)^{k+1}\lambda_{n-k}\right)a_n$$
$$\ge \left(\prod_{i=1}^{n-1}\left(\sqrt{a_i a_{i+1}}\right)^{2\left(\sum_{k=1}^{i}(-1)^{k+1}\lambda_{i-k+1}\right)}\right)\cdot a_n^{\left(1-\sum_{i=1}^{n-1}\lambda_k+(-1)^{k+1}\lambda_{n-k}\right)}$$
$$= \left(\prod_{i=1}^{n-1}(a_i)^{\lambda_i}\right)\cdot a_n^{\left(1-\sum_{i=1}^{n-1}\lambda_i\right)} = \left(\prod_{i=1}^{n}(a_i)^{\lambda_i}\right),$$

Where $\lambda_n = 1 - \sum_{i=1}^{n-1}\lambda_i$, and so

$$\sum_{i=1}^{n}\lambda_i a_i - \sum_{i=1}^{n-1}\lambda_i\left(\sqrt{a_i}-\sqrt{a_n}\right)^2 \le \prod_{i=1}^{n} a_i^{\lambda_i}$$

This ends the proof. ■

Remark. When comparing the inequality (3.1) with the inequality (1.3) and also the inequality (3.3) with the inequality (1.4), it is clearly seen that the inequality (3.1) is a refinement of the inequality (1.3), and the inequality (3.3) is a improvement of the inequality (1.4) by generalizing the number of terms to n numbers.

AUTHORS' CONTRIBUTIONS

Manuharawati: conceptualization, discussing with other authors to construct and prove theorems. Dian Savitri: discussing with other authors to construct and prove theorems, review. Muhammad Jakfar: discussing with other authors to construct and prove theorems, drafting and editing manuscrip

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